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A Unique Common Fixed Point Theorem under psi – varphi Contractive Condition in Partial Metric Spaces Using Rational Expressions

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Abstract

In this paper, we obtain a unique common fixed point theorem for two self maps satisfying psi - varphi contractive condition in partial metric spaces by using rational expressions.

Keywords: partial metric, weakly compatible maps, complete space.

1. Introduction

The notion of partial metric space was introduced by S.G.Matthews [3] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation ([2, 5, 7 -12], etc).

S.G.Matthews [3], Sandra Oltra and Oscar Valero[4] and Salvador Romaguera [6]and I.Altun, Ferhan Sola, HakanSimsek [1] prove fixed point theorems in partial metric spaces for a single map.

In this paper, we obtain a unique common fixed point theorem for two self mappings satisfying a generalized ψ -φ contractive condition in partial metric spaces by using rational expression.

First we recall some definitions and lemmas of partial metric spaces.

2. Basic Facts and Definitions

*Definition 2.1. [3].*A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all x, y, z $\in$ X:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \le p(x, y)$, $p(y, y) \le p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

(X, p) is called a partial metric space.

It is clear that $|p(x, y) - p(y, z)| \le p(x, z)$ for all x, y, z $\in$ X.

Also clear that $p(x, y) = 0$ implies $x = y$ from (p_1) and (p_2).

But if $x = y$, $p(x, y)$ may not be zero. A basic example of a partial metric space is the pair (R^+, p), where $p(x, y) = \max\{x, y\}$ for all $x, y \in R^+$.

Each partial metric p on X generates τ_0 topology τ_p on X which has a base the family of open p - balls $\{B_p(x, \varepsilon) / x \in X, \varepsilon > 0\}$ for all $x \in X$ and $\varepsilon > 0$, where $B_p(x, \varepsilon)= \{y \in X / p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to R^+$ given by

$$P^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{2.1}$$

is a metric on X.

*Definition 2.2. [3]***.** Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X, p) is said to converge to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in (X, p) is said to be Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and is finite .

(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, w.r.to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{m,n\to\infty} p(x_m, x_n)$.

Lemma 2.3. [3].Let (X, p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore,

$\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{m,n\to\infty} p(x_m, x_n)$.

3. Main Result

Theorem 3.1. Let (X, p) be a partial metric space and let $T, f : X \to X$ be mappings such that

(i) $\psi(p(Tx, Ty)) \le \psi(M(x, y)) - \varphi(M(x, y))$, $\forall\ x, y \in X$,

where $M(x, y) = \max\left\{p(fy,Ty)\dfrac{1+p(fx,Tx)}{1+p(fx,fy)},\ p(fx,fy)\right\}$ and $\psi: [0, \infty) \to [0, \infty)$ is continuous, non-decreasing with $\psi(t) = 0$ if and only if $t = 0$ and $\varphi:[0, \infty) \to [0, \infty)$ is lower semi continuous with $\varphi(t) = 0$ if and only if $t = 0$,

(ii) $T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X and

(iii) the pair (f, T) is weakly compatible.

Then T, f have a unique common fixed point of the form α in X.

Proof: Let $x_0 \in X$. From (ii), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = fx_{n+1} = Tx_n$, $n = 0, 1, 2,3, \cdots$

Case(a): Suppose $y_n = y_{n+1}$ for some n.

Then $fz = Sz$, where $z = x_{n+1}$. Denote $fz = Tz = \alpha$.

$$\begin{aligned}\psi(p(T\alpha, \alpha)) &= \psi(p(T\alpha, Tz))\\ &\le \psi(M(\alpha, z)) - \varphi(M(\alpha, z)).\end{aligned}$$

$$M(\alpha, z) = \max\left\{p(fz,Tz)\frac{1+p(f\alpha,T\alpha)}{1+p(f\alpha,fz)},\ p(f\alpha,fz)\right\}$$

$$= \max\left\{p(\alpha,\alpha)\frac{1+p(T\alpha,T\alpha)}{1+p(T\alpha,\alpha)},\ p(T\alpha,\alpha)\right\}$$

$$= p(T\alpha, \alpha), \qquad \text{from } (p_2).$$

Therefore

$$\psi(p(T\alpha, \alpha)) \le \psi(p(T\alpha, \alpha)) - \varphi(p(T\alpha, \alpha))$$
$$< \psi(p(T\alpha, \alpha)).$$

It is a contradiction.

Hence $f\alpha = T\alpha = \alpha$.

Hence α is a common fixed point of f and T.

Let β be another common fixed point of f and T such that $\alpha \ne \beta$.

$$\psi(p(\alpha, \beta)) = \psi(p(T\alpha, T\beta))$$

$$\le \psi\left(\max\left\{p(\beta,\beta)\frac{1+p(\alpha,\alpha)}{1+p(\alpha,\beta)},p(\alpha,\beta)\right\}\right) - \varphi\left(\max\left\{p(\beta,\beta)\frac{1+p(\alpha,\alpha)}{1+p(\alpha,\beta)},p(\alpha,\beta)\right\}\right)$$

$$= \psi(p(\alpha, \beta)) - \varphi(p(\alpha, \beta)), \qquad \text{from } (p_2)$$
$$< \psi(p(\alpha, \beta)).$$

It is a contradiction. Hence $\alpha = \beta$.

Thus α is the unique common fixed point of T and f.

Case(b): Suppose $y_n \ne y_{n+1}$ for all n.

$$\psi(p(y_n, y_{n+1})) = \psi(p(Tx_n, Tx_{n+1}))$$
$$\le \psi(M(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1}))$$
$$= \psi(\max\{p(y_n, y_{n+1}), p(y_n, y_{n-1})\}) - \varphi(\max\{p(y_n, y_{n+1}), p(y_n, y_{n-1})\}).$$

If $p(y_n, y_{n+1})$ is maximum, then

$$\psi(p(y_n, y_{n+1})) \le \psi(p(y_n, y_{n+1})) - \varphi(p(y_n, y_{n+1}))$$
$$< \psi(p(y_n, y_{n+1})).$$

It is a contradiction.

Hence $p(y_n, y_{n-1})$ is maximum.

Therefore

$$\psi(p(y_n, y_{n+1})) \le \psi(p(y_n, y_{n-1})) - \varphi(p(y_n, y_{n-1})) \tag{3.1}$$
$$\le \psi(p(y_n, y_{n-1})).$$

Since ψ is non – decreasing, we have $p(y_n, y_{n+1}) \le p(y_n, y_{n-1})$

Similarly $p(y_{n+2}, y_{n+1}) \le p(y_n, y_{n+1})$.

Thus $p(y_n, y_{n+1}) \le p(y_n, y_{n-1})$, n = 1, 2, 3, …

Thus $\{ p(y_n, y_{n+1}) \}$ is a non - increasing sequence of non-negative real numbers and must converge to a real number, say, $L \ge 0$.

Letting $n \to \infty$ in (3.1), we get

$\psi(L) \le \psi(L) - \varphi(L)$ so that $\varphi(L) \le 0$. Hence L = 0.

Thus

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0. \tag{3.2}$$

Hence from (p_2),

$$\lim_{n\to\infty} p(y_n, y_n) = 0. \tag{3.3}$$

By definition of p^s, we have $p^s(y_n, y_{n+1}) \le 2p(y_n, y_{n+1})$.

From (3.2), we have

$$\lim_{n\to\infty} p^s(y_n, y_{n+1}) = 0. \tag{3.4}$$

Now we prove that $\{y_n\}$ is Cauchy sequence in metric space (X, p^s). On contrary suppose that $\{y_n\}$ is not Cauchy. Then there exists an $\varepsilon > 0$ for which we can find two subsequences $\{y_{m(k)}\}$, $\{y_{n(k)}\}$ of $\{y_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$,

$$p^s(y_{m(k)}, y_{n(k)}) \ge \varepsilon \tag{3.5}$$

and

$$p^s(y_{m(k)}, y_{n(k)-1}) < \varepsilon. \tag{3.6}$$

From (3.5) and (3.6),

$$\begin{aligned}\varepsilon &\le p^s(y_{m(k)}, y_{n(k)})\\ &\le p^s(y_{m(k)}, y_{n(k)-1}) + p^s(y_{n(k)-1}, y_{n(k)})\\ &< \varepsilon + p^s(y_{n(k)-1}, y_{n(k)})\end{aligned}$$

Letting $k \to \infty$ and using (3.2), we have

$$\lim_{k\to\infty} p^s(y_{m(k)}, y_{n(k)}) = \varepsilon. \tag{3.7}$$

From (p_2), we get

$$\lim_{k\to\infty} p(y_{m(k)}, y_{n(k)}) = \frac{\varepsilon}{2}. \tag{3.8}$$

Letting $k \to \infty$ and using (3.2) and (3.7) in the inequality

$$| p^s(y_{m(k)-1}, y_{n(k)}) - p^s(y_{m(k)}, y_{n(k)}) | \le p^s(y_{m(k)}, y_{m(k)-1})$$

we get

$$\lim_{k\to\infty} p^s(y_{m(k)-1}, y_{n(k)}) = \varepsilon. \tag{3.9}$$

From (p_2), we get

$$\lim_{k\to\infty} p(y_{m(k)-1}, y_{n(k)}) = \frac{\varepsilon}{2}. \tag{3.10}$$

Letting $k \to \infty$ and using (3.2) and (3.7) in the inequality

$$| p^s(y_{m(k)}, y_{n(k)+1}) - p^s(y_{m(k)}, y_{n(k)}) | \le p^s(y_{n(k)}, y_{n(k)+1})$$

we get

$$\lim_{k\to\infty} p^s(y_{m(k)}, y_{n(k)+1}) = \varepsilon. \tag{3.11}$$

From (p_2), we get

$$\lim_{k\to\infty} p(y_{m(k)}, y_{n(k)+1}) = \frac{\varepsilon}{2}. \tag{3.12}$$

$$\begin{aligned}\psi(p(y_{m(k)}, y_{n(k)+1})) &= \psi(p(Tx_{m(k)}, Tx_{n(k)+1}))\\ &\le \psi(M(x_{m(k)}, x_{n(k)+1})) - \varphi(M(x_{m(k)}, x_{n(k)+1}))\\ &= \psi\left(\max\left\{p(y_{m(k)}, y_{m(k)+1})\frac{1+p(y_{m(k)-1}, y_{m(k)})}{1+p(y_{m(k)-1}, y_{n(k)})}, p(y_{m(k)-1}, y_{n(k)})\right\}\right)\\ &\quad - \varphi\left(\max\left\{p(y_{m(k)}, y_{m(k)+1})\frac{1+p(y_{m(k)-1}, y_{m(k)})}{1+p(y_{m(k)-1}, y_{n(k)})}, p(y_{m(k)-1}, y_{n(k)})\right\}\right).\end{aligned}$$

Letting $k \to \infty$ and using (3.12), (3.2), (3.10) and (3.8), we get

$$\psi\left(\frac{\varepsilon}{2}\right) \le \psi\left(\frac{\varepsilon}{2}\right) - \varphi\left(\frac{\varepsilon}{2}\right) < \psi\left(\frac{\varepsilon}{2}\right)$$

It is a contradiction.

Hence $\{y_n\}$ is Cauchy sequence in (X, p^s).

Thus $\lim_{m,n\to\infty} p^s(y_n, y_m) = 0$.

By definition of p^s and from (3.2), we get

$$\lim_{m,n\to\infty} p^s(y_n, y_m) = 0. \tag{3.13}$$

Suppose f(X) is complete.

Since $\{y_n\} \subseteq f(X)$ is a Cauchy sequence in the complete metric space $(f(X), p^s)$, it follows that $\{y_n\}$ converges in $(f(X), p^s)$.

Thus $\lim_{n\to\infty} p^s(y_n, \alpha) = 0$ for some $\alpha \in f(X)$.

There exists $w \in X$ such that $\alpha = fw$.

Since $\{y_{n+1}\}$ is Cauchy in X and $\{y_n\} \to \alpha$, it follows that $\{y_{n+1}\} \to \alpha$.

From Lemma 2.3(b) and (3.13), we have

$$p(\alpha, \alpha) = \lim_{n\to\infty} p(y_n, \alpha) = \lim_{n\to\infty} p(y_{n+1}, \alpha) = \lim_{m,n\to\infty} p(y_n, y_m) = 0. \tag{3.14}$$

$$\begin{aligned}p(Tw, \alpha) &\le p(Tw, Tx_{n+1}) + p(Tx_{n+1}, \alpha) - p(Tx_{n+1}, Tx_{n+1})\\ &\le p(Tw, Tx_{n+1}) + p(Tx_{n+1}, \alpha).\end{aligned}$$

Letting $n\to\infty$, we have

$$p(Tw, \alpha) \le \lim_{n\to\infty} p(Tw, Tx_{n+1}).$$

Since ψ is is continuous and non - decreasing, we have

$$\psi(p(Tw, \alpha)) \le \lim_{n\to\infty} \psi(p(Tw, Tx_{n+1}))$$

$$= \lim_{n\to\infty} \psi\left(\max\left\{ p(y_n, y_{n+1}) \frac{1+p(\alpha, Tw)}{1+p(\alpha, y_n)}, p(\alpha, y_n) \right\} \right)$$

$$- \lim_{n\to\infty} \varphi\left(\max\left\{ p(y_n, y_{n+1}) \frac{1+p(\alpha, Tw)}{1+p(\alpha, y_n)}, p(\alpha, y_n) \right\} \right)$$

$= \psi(0) - \varphi(0) = 0.$

It follows that Tw = α. Thus Tw = α = fw.

Since the pair (f, T) is weakly compatible, we have fα = Tα.

As in Case(a), it follows that α is the unique common fixed point of T and f.

Example 3.2. Let X = [0, 1] and p(x, y) = max{x, y} for all x, y $\in$ X . Let T, f : X → X, f(x) = x/3 and T(x) = $x^2/6$, ψ : [0,∞) → [0,∞) by ψ(t) = t and φ: [0,∞) → [0,∞) by φ(t) = t/2 . Then the conditions (ii) and (iii) are satisfied and

$\psi(p(Tx, Ty)) = p(Tx, Ty)$

$$= \max\left\{\frac{x^2}{6}, \frac{y^2}{6}\right\}$$

$$\leq \max\left\{\frac{x}{6}, \frac{y}{6}\right\}$$

$$= \frac{1}{2} p(fx, fy) \leq \frac{1}{2} \max \left\{ p(fy, Ty)\frac{1+p(fx, Tx)}{1+p(fx, fy)},\ p(fx, fy)\right\}$$

$$= \psi(M(x, y)) - \varphi(M(x, y)).$$

Where $M(x, y) = \max \left\{ p(fy, Ty)\frac{1+p(fx, Tx)}{1+p(fx, fy)},\ p(fx, fy)\right\}$

Thus (i) is holds.

Hence all conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of T and f.

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